No. 16

THE

MATHEMATICAL GAZETTE.

EDITED BY

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WITH THE CO-OPERATION OF

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LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,

AND BOMBAY.

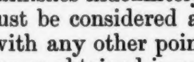
UNIFORMLY ACCELERATED MOTION.

As I have been, for some few years past, in the habit of showing my pupils a proof of the formulae for uniformly accelerated motion very like that given by Mr. Dobbs in the October number of the *Mathematical Gazette,* I venture to add the few following remarks to his.

The matter seems best treated, from the vector point of view, as follows:

1. When a point is moving in any manner, the ratio (*displacement in a given interval* τ)/τ is called the average velocity for that interval. Like displacement, it is a vector.

2. The *velocity at any instant* is that vector, if any, to which the average velocity for an interval containing that instant approaches as its limit, when the interval diminishes indefinitely. It is necessary to note that the instant must be considered as capable of coinciding with either end of, or with any other point of time in, the interval; if the limiting vector so obtained is not the same for all positions of the instant in the interval, there is, properly speaking, no instantaneous velocity.



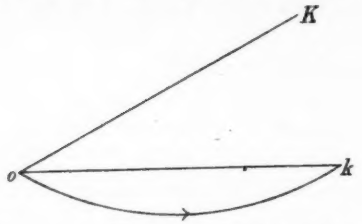
If, then, *op*, *oq* represent the average velocity for any interval, and the instantaneous velocity at any instant of the interval, as the interval diminishes indefinitely, so also does the vector *pq*.

[For further amplification, see Clifford, *Kinematic*, vol. I., pp. 47-48, 56-58.]

3. The following proposition places many kinematical discussions without trouble on a rigorous basis.

If a point has at each instant of its motion an instantaneous velocity, *its path may be represented to any required degree of accuracy by supposing the point to travel for a short interval* τ

with the velocity it has at the beginning of the interval, for another short interval τ with the velocity it has at the beginning of this second interval, and so on.



Distinguishing vector quantities by a bar drawn over them, let $\bar{v}_1, \bar{v}_2, \bar{v}_3, \dots$ be the velocities at the beginnings of the intervals τ, and $V_1, V_2, V_3, \dots$ the average velocities for each interval; o the initial position, k the final position of the point after n such intervals; K the final position of the point if it were displaced on the above hypothesis; t the time of making the displacement o to k. $\therefore \quad t = n\tau.$

Then (Definition of Average Velocity),

$$\overline{oK} = \overline{V}_1 . \tau + \overline{V}_2 . \tau + \overline{V}_3 . \tau + \dots$$
$$= (\overline{V}_1 + \overline{V}_2 + \overline{V}_3 + \dots)\tau$$
$$\overline{ok} = (\bar{v}_1 + \bar{v}_2 + \bar{v}_3 + \dots)\tau$$
$$\therefore \quad \overline{kK} = [(\overline{V}_1 - \bar{v}_1) + (\overline{V}_2 - \bar{v}_2) + (\overline{V}_3 - \bar{v}_3) + \dots]\tau.$$

Now let $\bar{u}$ be the greatest of the vectors $\overline{V}_1 - \bar{v}_1, \overline{V}_2 - \bar{v}_2 \dots$.

$\therefore$ kK is numerically less than $nu\tau$ or ut.

But u, and therefore ut, diminishes indefinitely with τ (§ 2). The proposition is therefore established.

[The same may also be proved by means of a figure.]

Corollary. The same is evidently true if the point is supposed to travel throughout each interval with the instantaneous velocity at the end of the interval, or indeed with the instantaneous velocity at any assigned instant in the interval.

4. The above proposition, taking into account the fact that vector addition is commutative, gives the principle of the Independence of Displacements in a rigorous form.

Then, for the uniform acceleration formula, taking Mr. Dobbs's first figure, and dividing the time t into $2n$ intervals each equal to τ, let op be the initial velocity, pq the hodograph for time t, $op, op', op'' \dots$ the velocities at the beginnings $\dots oq'', oq', oq \dots$ the velocities at the ends of intervals τ taken in succession, the first series from the beginning, the latter from the end, of the time t.

The mean of the vectors $\overline{op}, \overline{oq}, \overline{op'}, \overline{oq'}$, etc. ... is $\overline{or}$, r being the mid-point of pq.

Hence the whole displacement is equivalent to

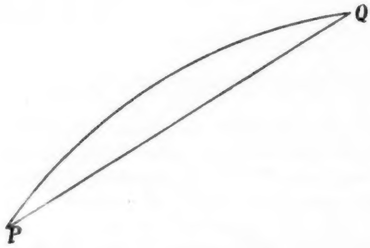
$$2n . \overline{or} . \tau = \overline{or} . t = (\bar{u} + \tfrac{1}{2}\bar{f} . t)t,$$

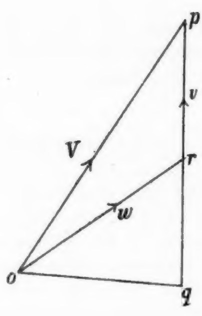
$\bar{f}$ denoting the acceleration. § 3 shows this is also the displacement when τ is diminished indefinitely.

From the fact that $\overline{or}$ is the average velocity for the interval t, many properties of the parabola may be proved kinematically.

5. The proof of $v^2 = u^2 \pm 2fs$, as I have been accustomed to give it, does not differ materially from that of Mr. Dobbs's *Corollary IV.* It is well, however, to note that if the positive sign is to be taken in all cases, s (the resolved displacement in the direction of $\bar{f}$) must be reckoned positive in the sense of $\bar{f}$, negative in the opposite sense.

6. From the fact that $\overline{or}$ is the average velocity for the interval t follows an easy discussion of the range of a projectile on an inclined plane.





Let PQ denote the range, op, oq the velocities at P, Q respectively; pq (vertical) is the hodograph; $pr = qr = \frac{gt}{2}$, where t is the time of flight.

Then, since or is the mean velocity, the range

$$PQ = or \,.\, t = \frac{2 \,.\, or \,.\, rp}{g} = \frac{2uv}{g},$$

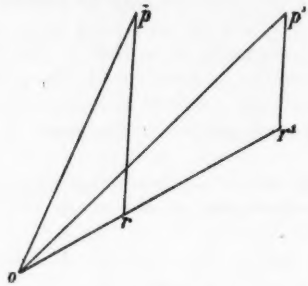
where u, v are the *oblique components* of $V(=op)$ the velocity of projection, parallel to the plane and the vertical respectively.

Further the angle orp is constant.

$\therefore$ $or \,.\, rp \propto$ area of triangle orp; and if $V(=op)$ be constant in magnitude, the area of this triangle is a maximum when $or = rp$, or $v = u$.

Hence the maximum range for a given speed is obtained by taking the direction of projection equally inclined to the vertical and the plane.

Next, for a given value of the range, and of the initial speed, we can obtain two triangles *orp*, *or′p′* of the required area, and they will be such that $or = p'r'$, $pr = or'$.



Hence, for any range short of the greatest, there are two directions of projection equally inclined to the direction of greatest range. H. A. ROBERTS.

A THEOREM OF ISOPERIMETRIC LOOPS.

(1) The following note is intended to draw attention to a curious relation which I arrived at, some years ago, between the lengths of the common tangents of two *intersecting* ellipses, and the lengths of the arcs between their points of contact.

The property proved may be regarded as an extension of Graves' Theorem [SALMON; Art. 399] and is deduced from that theorem in conjunction with a lemma proved in this note.

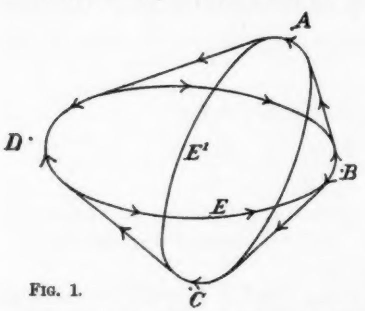


FIG. 1.

In fig. 1 it will be seen that there are two endless loops marked out by arrows and each free from angularities. For distinctness those which mark out one loop are drawn circulating in a sense contrary to those marking out the other.

A, B, C, D are the angular points of the (convex) common tangent quadrilateral fig. 1. In fig. 2 we have the second case in which this quadrilateral is re-entrant.

Having given that

$$\left.\begin{aligned} &AB + AD = CB + CD \text{ in case (1)} \\ \text{or}\quad &AB \sim AD = CB \sim CD \text{ in case (2)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(a)$$

it is required to show that the endless loops above-mentioned are of equal length.

Let T denote the *sum* of the two tangents from A to one ellipse; t to the other ($T>t$).

Let S s be the lengths of the arcs between their respective points of contact, and suppose the same letters carrying accents to denote corresponding quantities related to the point C. Then E being the entire circumference of the ellipse farthest from A and C, fig. 1, we have evidently for the lengths of the loops

$$(T-t)+s+(E-S)$$

and

$$(T'-t')+s'+(E-S').$$

Assuming now that under the conditions (a) we can draw through A and C two *ellipses* confocal with the given ones, each with each, we have by Graves' Theorem:

$$T-S=T'-S'$$

and

$$t-s=t'-s'$$

whence it follows at once that the loops are of equal length as was to be proved.

PROOF OF THE LEMMA.

(2) *This refers to fig.* 2 *alone*, but with slight modifications (which to save space are omitted here) it applies also to fig. 2'.

Take fig. 2, $AH=a'$; $CH=b'$; $CS=c'$; $AS=d'$; the angle $DAB=2A$; $SAH=2A'$; $BCD=2C$; $SCH=2C'$, where S, H are the foci of *any* ellipse inscribed to A B C D.

By hypothesis

$$b+c=a+d$$

$$\therefore (b+c+BD)(b+c-BD)=(a+d+BD)(a+d-BD)$$

or

$$4bc \,.\, \cos^2 C=4ad \,.\, \cos^2 A \text{.......................} (1)$$

Again, $bc/b'c'=\sin BHC \operatorname{cosec} HBC \,.\, \sin DSC \operatorname{cosec} SDC.$

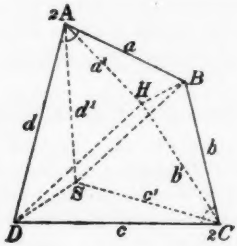


FIG. 2.

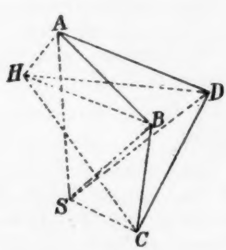


FIG. 2'.

Now by a known property AS, AH, etc., are equally inclined to the adjacent sides of the quadrilateral, and any pair of opposite sides subtend supplementary angles at S or at H.

Hence $bc/b'c' = \sin AHD \operatorname{cosec} SBA \,.\, \sin ASB \operatorname{cosec} HDA$
$= \sin AHD \operatorname{cosec} HDA \,.\, \sin ASB \operatorname{cosec} SBA$
$= ad/a'd'$;

$\therefore$ by (1) $4b'c' \cos^2 C = 4a'd' \cos^2 A$ ....(2)

Again, the square of the semi-axis minor of the ellipse equals the product of the perpendiculars from S, H, on *any* of the sides of the quadrilateral.

Thus $b'c' \sin (C+C') \sin (C-C') = a'd' \sin (A+A') \sin (A-A')$

or, $b'c' (\cos^2 C' - \cos^2 C) = a'd' (\cos^2 A' - \cos^2 A)$.

Therefore by (2) $4\, b'c' \cos^2 C' = 4\, a'd' \cos^2 A'$

from which we have immediately

$$(b'+c'+SH)(b'+c'-SH) = (a'+d'+SH)(a'+d'-SH)$$

that is, $(b'+c')^2 - SH^2 = (a'+d')^2 - SH^2$

or, $b'+c' = a'+d'$,

showing that an *ellipse* with foci S, H, can be drawn through the points A and C, which proves the lemma assumed in the proof of the theorem.

It will be noticed that the steps of this demonstration are reversible.

(3) The conditions (a) may both be included in a single statement:

(b) *The sides of the common tangent quadrilateral all touch a circle external to both ellipses.*

This is a known theorem of elementary geometry and it may also be inferred from the fact that by (a) AB, AD and CB, CD fig. 2, 2′ are four focal radii to two points A, C on the same conic. Now these all touch a circle whose centre is the pole of the chord AC of this conic.

(4) Other transformations of (a) can be given but it would be impossible in a short space to treat these satisfactorily. However we merely mention some which may be of interest. Referring to the ellipses as $E\ E'$,

(c) *The foci of E lie on a conic confocal with E' and vice versâ.* And again,

(d) *Two conics $F\ F'$ can be drawn; F having double contact with E and confocal with E'; F' having double contact with E' and confocal with E.* It may be remarked that $F\ F'$ *may* both be ellipses but that *one of these must be an ellipse.* Lastly, the common pole of the chords of contact of EF, $E'F'$ is the centre of the circle in (b).

N.B.—Particular cases occur when point-pairs are substituted for one or both ellipses EE'. C. E. McVicker.

MATHEMATICAL NOTES.

69. *In a spherical quadrangle the arcs joining the mid-points of the three pairs of opposite sides are concurrent.* (Question 90, p. 16. For Geometrical Solution *v.* note 68, p. 214, vol. I.)

Let A, B, C, D be the four vertices of the quadrilateral L, M, N, L' M' N' the middle points of the arcs BC, CA, AB, AD, BD, CD. Let the arcs LL', MM' meet at K.

Let the great circles of which MN, BC are arcs meet at P_1, P_1'; then P_1MNP_1' is a semicircle. In the triangle ANM

$$\sin AN \sin ANM = \sin AM \sin AMN. \qquad \text{.........................(1)}$$

In the triangles NBP_1', MCP_1

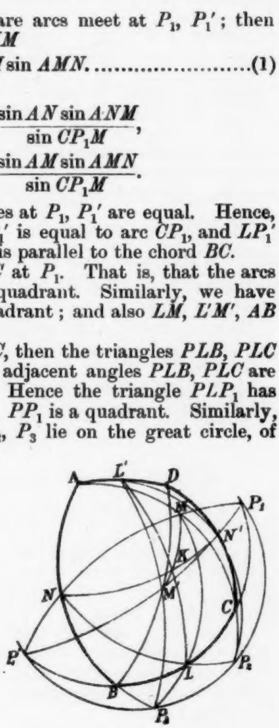
$$\sin BP_1' = \frac{\sin BN \sin BNP_1'}{\sin BP_1'N} = \frac{\sin AN \sin ANM}{\sin CP_1M},$$

$$\sin CP_1 = \frac{\sin CM \sin CMP_1}{\sin CP_1M} = \frac{\sin AM \sin AMN}{\sin CP_1M}.$$

$\therefore$ by (1), $\sin BP_1' = \sin CP_1$, since the angles at P_1, P_1' are equal. Hence, since each is less than a semicircle, arc BP_1' is equal to arc CP_1, and $LP_1' = LP_1 =$ a quadrant; and the diameter P_1P_1' is parallel to the chord BC.

Similarly, it is shown that $M'N'$ cuts BC at P_1. That is, that the arcs MN, $M'N'$, BC are concurrent, and LP_1 a quadrant. Similarly, we have NL, $N'L'$, CA meeting at P_2, and MP_2 a quadrant; and also LM, $L'M'$, AB meeting at P_3, and NP_3 a quadrant.

Let P* be the pole of the small circle ABC, then the triangles PLB, PLC have their sides respectively equal; $\therefore$ the adjacent angles PLB, PLC are equal, and each of them is a right angle. Hence the triangle PLP_1 has PLP_1 a right angle, and LP_1 a quadrant; $\therefore$ PP_1 is a quadrant. Similarly, PP_2 and PP_3 are quadrants. Hence P_1, P_2, P_3 lie on the great circle, of which P is a pole.†



Again, the arc LP_3M cuts the sides of the triangle P_2NP_1;

$$\therefore \quad \frac{\sin NM}{\sin NL} = \frac{\sin NLM}{\sin NML},$$

$$\frac{\sin P_1P_3}{\sin P_1M} = \frac{\sin P_1MP_3}{\sin P_1P_3M},$$

$$\frac{\sin P_2L}{\sin P_2P_3} = \frac{\sin P_2P_3L}{\sin P_2LP_3};$$

$\therefore$ by multiplication and reduction,

$$\frac{\sin NM}{\sin P_1M} \cdot \frac{\sin P_1P_3}{\sin P_2P_3} \cdot \frac{\sin P_2L}{\sin NL} = 1.$$

Similarly, the sides of the triangle P_2P_1N' are cut by the arc $L'P_3M'$, and

$$\frac{\sin P_1M'}{\sin N'M'} \cdot \frac{\sin N'L'}{\sin P_2L'} \cdot \frac{\sin P_2P_3}{\sin P_1P_3} = 1,$$

also the triangle $N'NP_2$ is cut by the arc KLL', and

$$\frac{\sin N'K}{\sin NK} \cdot \frac{\sin NL}{\sin P_2L} \cdot \frac{\sin P_2L'}{\sin N'L'} = 1.$$

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* Not shown in the figure.

† Or this may be inferred from the fact that the diameters $P_1'P_1$, $P_2'P_2$, $P_3'P_3$ are respectively parallel to the chords BC, CA, AB, and are therefore in a plane parallel to the plane ABC.

Multiplying these results and reducing, we have

$$\frac{\sin NM}{\sin P_1M} \cdot \frac{\sin P_1M'}{\sin N'M'} \cdot \frac{\sin N'K}{\sin NK} = 1. \qquad \text{.........................(2)}$$

Let the arc MM' meet the arc NN' at K'. Then the triangle $N'NP_1$ is cut by the arc $MM'K'$, and we have

$$\frac{\sin NM}{\sin P_1M} \cdot \frac{\sin P_1M'}{\sin N'M'} \cdot \frac{\sin N'K'}{\sin NK'} = 1.$$

Hence, by (2),

$$\frac{\sin N'K}{\sin NK} = \frac{\sin N'K'}{\sin NK'}.$$

Therefore K and K' coincide since the arcs involved are each less than a semicircle. That is, the arcs LL', MM', NN' are concurrent.

COR. From the above demonstration it is evident that if two spherical triangles are so related that the points of intersection of corresponding sides lie on a great circle, then the arcs joining the corresponding angles meet in a point. The converse readily follows. J. C. PALMER.

70. *A geometrical method of trisecting any angle with the aid of a rectangular hyperbola.* (C. V. DURELL and W. F. BEARD.)

Let $B\hat{O}C$ be the given angle. From centre O describe a circle, cutting BO produced in A.

Bisect OA at D. Describe a rectangular hyperbola with centre D touching OC at O. Let this hyperbola cut the circle at P.

Join OP.

Then $\because$ OC is a tangent $P\hat{O}C = P\hat{A}O$ (for in rectangular hyperbola any chord subtends equal or supplementary angles at the ends of a diameter).

$$\therefore \; B\hat{O}P = 2O\hat{A}P = 2P\hat{O}C;$$

$$\therefore \; P\hat{O}C = \tfrac{1}{3}B\hat{O}C;$$

$$\therefore \; OP \text{ is a trisector of } B\hat{O}C.$$

To describe this hyperbola,

Draw a circle centre O radius OD, cutting OC and the $\perp$ at O to OC in T, t, G, g. Then DT, Dt are the asymptotes, and DG, Dg the axes of the required hyperbola.

Cut off DS so that $DS^2 = DT \cdot Dt$, and DH so that $DH^2 = \tfrac{1}{2}DS^2$.

Then H and S are a vertex and focus of the hyperbola.

We can then mechanically describe the hyperbola either by the method with foci, or by Cunynghame's method (*vide* Taylor's *Ancient and Modern Geometry*, p. 177).

EXAMINATION QUESTIONS AND PROBLEMS.

Our readers are earnestly asked to help in making this section of the GAZETTE *attractive by sending either original or selected problems.*

Solutions should be sent within three months of the date of publication. They should be written clearly on one side of the paper. Contractions not intended for printing should be avoided. Figures should be drawn with the greatest care on as small a scale as possible, and on a separate sheet.

The question need not be re-written, but the number should precede every solution.

The source of problems when not otherwise indicated is shown by —C. (Cambridge), O. (Oxford), D. (Dublin), W. (Woolwich), Sc. (Science and Art Department).

280. The angles of one face of the in-cube of a sphere meet the surface at A, B, C, D. If P is any point on the sphere

$$3\Sigma\cos^2 PA = 4.$$ W. S. COONEY.

281. SY is the perpendicular from the pole on the tangent at P to a given curve. Bisect SP in Q; join YQ and produce it to O so that $QO : YO =$ chd. of curv. at P through $S : 4SP$. Then O is the centre of curvature at Y to the pedal. R. F. DAVIS.

282. Two circles are described with the foci of the ellipse as centres, so that the sum of the squares of their radii is a^2+b^2 (usual notation). The circles cut any tangent to the ellipse in a harmonic range. (B.A., London.) F. A. FIELD.

283. A and B are two squads selected from n soldiers, so that (i.) A is not to contain fewer soldiers than B, nor more than a; (ii.) B is not to contain more than b soldiers where $b \gtreqless a$. If the men are undistinguishable, find the number of arrangements.
P. A. MACMAHON.

284. LMN is the Simson line of the triangle ABC with respect to P, on the side of BC remote from A. PA, PB, PC subtend angles A', B', C' at the circumference. Show that

(1) $\sin^2 A' = \lambda u/vw$, etc., where u, v, w denote $MN \operatorname{cosec} A$,

$NL \operatorname{cosec} B$, $LM \operatorname{cosec} C$, and $\lambda\left(-\dfrac{\sin A}{u}+\dfrac{\sin B}{v}+\dfrac{\sin C}{w}\right)=\Pi \sin A$;
(Trip., 1891.)

(2) $PL \,.\, PA = PM \,.\, PB = PN \,.\, PC$;

(3) $$\sin^2 A \,.\, \sin^2 A' \,.\, \frac{LM \,.\, LN}{MN} = \sin^2 B \,.\, \sin^2 B' \,.\, \frac{MN \,.\, ML}{NL}$$

$$= \sin^2 C \,.\, \sin^2 C' \,.\, \frac{NL \,.\, NM}{LM}.$$

W. J. GREENSTREET.

285. Let l be the recurring period of r digits of the decimal equivalent to the vulgar fraction $\dfrac{1}{n}$. Let $n\kappa = p \,.\, 10^q - 1$, where κ is a number of 9 digits $(9 < r)$. Show that (1) l terminates with the group κ; (2) the remaining digits may be found by continuously multiplying by p.

[Thus, $29 \times 31 = 900 - 1$. The period for $\frac{1}{29}$ terminates with 31, and the remaining digits of the period can be found by multiplying continuously by 9.] E. M. LANGLEY.

286. $\Sigma \sin^{-1} bc/\sqrt{(a^2+b^2)(a^2+c^2)}=\pi/2$. A. LODGE.

287. All the quantities being real, prove that the limits of

$$\left[\sum_{r=1}^{p-1} x^2_{qr} - \sum_{r=1}^{p-2} x_{qr}x_{qr+q}\right] \Big/ \left[\sum_{r=1}^{pq-1} x_r^2 - \sum_{r=1}^{pq-2} x_r x_{r+1}\right]$$

are o and q. F. S. MACAULAY.

288. A, B, C, D are four boxes arranged in a circle, and n letters are divided into four packets, each of which may contain any number (zero included) of letters. A packet is placed in each box so that no box contains more letters than are in box A or fewer letters than are in box C. Find the number of arrangements. P. A. MACMAHON.

289. Find an indefinite number of rational right-angled triangles whose legs differ by unity. [Use

$$(2a+b+2c)^2+(a+2b+2c)^2=(2a+2b+3c)^2 \text{ if } a^2+b^2=c^2.]$$

ARTEMAS MARTIN.

290. A circle is inscribed in a quadrant of a circle, and a second circle is drawn touching this circle, the quadrant, and one radius of the quadrant. Prove that the radius of the second is $\frac{1}{7}$th of the distance of its centre from the other radius of the quadrant.

C. E. McVICKER.

291. There are 10 tickets:—5 blanks, and the others marked 1, 2, 3, 4, 5. What is the probability of drawing 10 in 3 trials (a) tickets replaced; (b) tickets not replaced. (Actuaries, Ed., 1891.) R. F. MUIRHEAD.

292. If $f(x)=a_0+a_1x+a_2x^2+\dots$ be a series whose sum is known, show that the sum of every nth term beginning with the mth, ($m \not> n$) is $\frac{1}{n}\sum_{r=n}^{r=1}[f(\omega_r x)\times\omega_r^{n-m+1}]$, where $\omega_1, \omega_2, \dots \omega_n$ are the nth roots of unity. Apply this to sum the series

$$(a)\ \frac{x^2}{2}\sin 2\theta+\frac{x^5}{5}\sin 5\theta+\dots;\quad (b)\ 1-\tfrac{1}{3}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{9}-\dots.$$

T. ROACH.

293. If two random points are taken within a given triangle, show that their join is equally likely to cut any pair of sides of the triangle. C. SANDBERG.

294. Any two lines l, m, are drawn through the vertex A of the triangle ABC. BL parallel to m meets l in L; CM parallel to l meets m in M. BL cuts CM in N. A', B', C' are mid points of sides. Prove $A'N$, $B'M$, $C'L$ are concurrent. J. A. THIRD.

295. Prove that Christmas Day is least likely to fall on Monday or Saturday, and most likely on Sunday, Tuesday, or Friday; and find numbers proportional to the probabilities of these events. (Welsh Medal, Kingswood School.)

T. P. THOMPSON.

296. BC, AC are straight paths. A', B', two men, start from A and B and run with uniform speed towards C. As they run they come into a straight line with a post P, and again with a post Q; in each case the post is half-way between them. When B' has reached C they stop running, and walk at quick march to meet each other. Show (geometrically) that, when they meet, the posts P and Q will be in a straight line with them.

R. B. WORTHINGTON.

297. Find the locus of the centre of a sphere which rolls on a parabolic wire, touching it at two points. (C.)

298. Find the sum and product of the non-zero roots of

$$\Sigma \tan^{-1}(a-x) = \Sigma \tan^{-1} a = \tan^{-1} s. \qquad \text{(C.)}$$

299. A bag contains a red, b yellow, and c blue balls. A ball is drawn and shown to X and Y. X, who speaks the truth once in x times, says it is blue. Y, who speaks the truth once in y times, says it is red. What is the consequent chance of its being yellow? (C.)

SOLUTIONS.

A great number of solutions are in hand, and will be published as sufficient space is available.

ERRATUM.

P. 215, l. 13: after 'another,' read: "similarly, $lnl'n'$ is a parallelogram, and ll', nn' bisect each other."

150. *If x, y, z be the coordinates of any point in space referred to rectangular axes, and r, r' the distances of x, y, z from any two fixed points, prove that the function $V \equiv a \log(br/r')$ satisfies the equations*

$$\frac{\partial}{\partial x}\left(\frac{1}{U}\frac{\partial V}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{U}\frac{\partial V}{\partial y}\right) = \frac{\partial}{\partial z}\left(\frac{1}{U}\frac{\partial V}{\partial z}\right) = \frac{1}{U}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right),$$

where
$$U \equiv \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2;$$

and that there is no solution more general. F. S. MACAULAY.

Solution by PROPOSER.

Put H for each of the four equals of the given equations, and

$$L \text{ for } \frac{1}{U}\frac{\partial V}{\partial x}, \quad M \text{ for } \frac{1}{U}\frac{\partial V}{\partial y}, \quad N \text{ for } \frac{1}{U}\frac{\partial V}{\partial z}.$$

Then
$$H = L_x = M_y = N_z = \frac{1}{U}(V_{xx} + V_{yy} + V_{zz}); \quad \text{.....................(1)}$$

and from the identity $U \equiv V_x^2 + V_y^2 + V_z^2$, we have

$$U = (L^2+M^2+N^2)\,U^2,$$

$$\therefore\; U = \frac{1}{L^2+M^2+N^2},\quad V_x = \frac{L}{L^2+M^2+N^2},\quad V_y = \frac{M}{L^2+M^2+N^2},$$

$$V_z = \frac{N}{L^2+M^2+N^2}. \qquad\qquad (2)$$

Hence,

$$V_{xx} = \frac{L_x}{L^2+M^2+N^2} - \frac{2L(LL_x+MM_x+NN_x)}{(L^2+M^2+N^2)^2}$$

$$= \frac{(M^2+N^2-L^2)H - 2L(MM_x+NN_x)}{(L^2+M^2+N^2)^2},\quad \text{since } L_x = H.$$

Hence,

$$H = \frac{1}{U}(V_{xx}+V_{yy}+V_{zz})$$

$$= \frac{(L^2+M^2+N^2)H - 2L(MM_x+NN_x) - 2M(NN_y+LL_y) - 2N(LL_z+MM_z)}{(L^2+M^2+N^2)}.$$

Here H cancels on the two sides of the equation, and we have

$$\frac{1}{L}(M_z+N_y) + \frac{1}{M}(N_x+L_z) + \frac{1}{N}(L_y+M_x) = 0. \qquad\qquad (3)$$

Again,

$$V_{yz} = \frac{\partial}{\partial y}\left(\frac{N}{L^2+M^2+N^2}\right) = \frac{\partial}{\partial z}\left(\frac{M}{L^2+M^2+N^2}\right) \text{ from (2);}$$

$$\therefore\; (L^2+M^2-N^2)N_y - 2N(LL_y+MM_y) = (N^2+L^2-M^2)M_z - 2M(NN_z+LL_z).$$

The term $-2MNM_y$ cancels out with $-2MNN_z$ from (1); hence

$$(M^2-N^2)(N_y+M_z) + L^2(N_y-M_z) - 2NLL_y + 2LML_z = 0.$$

This may be written

$$(M^2-N^2)(M_z+N_y) + LM(N_x+L_z) - NL(L_y+M_x)$$
$$- L[L(M_z-N_y) + M(N_x-L_z) + N(L_y-M_x)] = 0.$$

We have two other equations similar to this, and these, taken with (3), give

$$M_z+N_y=0,\quad N_x+L_z=0,\quad L_y+M_x=0, \qquad\qquad (4)$$

and
$$L(M_z-N_y) + M(N_x-L_z) + N(L_y-M_x) = 0,$$

or
$$LM_z + MN_x + NL_y = 0. \qquad\qquad (5)$$

The only other possibility is that $L^2+M^2+N^2=0$, a case which we shall not consider, since it would make V_x, V_y, V_z all infinite. There are other cases in which one or more of the quantities L, M, N vanish; but they do not lead to so general a solution as when no one of them vanishes.

The equations satisfied by L, M, N, H are then

$$H = L_x = M_y = N_z,$$

and equations (4) and (5).

Differentiate equations (4) with respect to x, y, z respectively; then

$$L_{yz} = M_{zx} = N_{xy} = 0. \qquad\qquad (6)$$

Differentiating the first of equations (4),

$$M_{yzz} + N_{zyy} = 0,$$

$$\therefore\; H_{zz} + H_{yy} = 0, \text{ etc.}$$

$$\therefore\; H_{xx} = H_{yy} = H_{zz} = 0. \qquad\qquad (7)$$

Also, $$H_{yz}=L_{xyz}=0, \text{ from (6)};$$
$$\therefore\ H_{yz}=H_{zx}=H_{xy}=0. \quad\text{.......(8)}$$

From (7) and (8) it follows that H is a linear function of x, y, z;
$$\therefore\ H=ax+by+cz+d. \quad\text{.......(9)}$$

We can now find the general values of L, M, N. We have
$$L_x=M_y=N_z=ax+by+cz+d \text{ from (1) and (9)};$$
$$M_{zx}=0 \text{ from (6)},\ M_{zy}=H_z=c,\ M_{zz}=-N_{yz}=-H_y=-b;$$
$$\therefore\ M_z=-N_y=cy-bz+f;$$
similarly, $$N_x=-L_z=az-cx+g;$$
$$L_y=-M_x=bx-ay+h.$$

These give us all the first differential coefficients of L, M, N with respect to x, y, z. We thus have, on integrating,
$$\left.\begin{aligned} L&=-\tfrac{1}{2}a(x^2+y^2+z^2)+x(ax+by+cz+d)+hy-gz+l,\\ M&=-\tfrac{1}{2}b(x^2+y^2+z^2)+y(ax+by+cz+d)+fz-hx+m,\\ N&=-\tfrac{1}{2}c(x^2+y^2+z^2)+z(ax+by+cz+d)+gx-fy+n.\end{aligned}\right\}\quad\text{.........(10)}$$

These values of L, M, N satisfy equations (1) and (4), but they have still to satisfy (5). By substitution it will be found that (5) is satisfied, provided
$$df=cm-bn,\quad dg=an-cl,\quad dh=bl-am.$$

The constants a, b, c, d, l, m, n may then be considered as arbitrary, and and f, g, h are known in terms of them.

We now know L, M, N, and therefore we also know V_x, V_y, V_z from (2). The function V can therefore be found, and it cannot involve more than 8 arbitrary constants, viz., a, b, c, d, l, m, n and an additive constant for integration. To continue this way of solving and bring V to the form $k\log(k'r/r')$ is troublesome; but it is sufficient to notice that this form for V also involves 8 arbitrary constants, viz., k, k' and three each in r, r', and it will be found on substitution that the function $k\log(k'r/r')$ does satisfy the given differential equations.

170. *The sum of the reciprocals of all positive primes is infinite.*

Solution by F. S. Macaulay.

If p stand successively for the primes 2, 3, 5 ... and n for all positive integers 1, 2, 3 ... we have
$$\Pi\left(1-\frac{1}{p}\right)^{-1}=\Pi\left(1+\frac{1}{p}+\frac{1}{p^2}+\ldots\right)$$
$$=\Sigma\frac{1}{p^\alpha q^\beta r^\gamma\ldots}=\sum_{n=1}\frac{1}{n}.$$

But $\Sigma\frac{1}{n}$ is divergent, therefore $\Pi\left(1-\frac{1}{p}\right)^{-1}$, which is the same as
$$\Pi\left(1+\frac{1}{p-1}\right),$$
is also divergent.
$$\therefore\ \Sigma\frac{1}{p-1} \text{ is divergent};$$
$$\therefore\ \Sigma\frac{1}{p} \text{ is divergent};$$
each term in $\Sigma\frac{1}{p}$ bearing a finite ratio to the corresponding term in $\Sigma\frac{1}{p-1}$.

194. *Prove that the complete solution of the equations,*

$$x\left(\frac{1}{y}+\frac{1}{z}\right)(y-z)^2=ax^3+bx^2+cx+d, \qquad (1)$$

$$y\left(\frac{1}{z}+\frac{1}{x}\right)(z-x)^2=ay^3+by^2+cy+d, \qquad (2)$$

$$z\left(\frac{1}{x}+\frac{1}{y}\right)(x-y)^2=az^3+bz^2+cz+d, \qquad (3)$$

can be found without solving any equation of higher order than the third; and that the number of solutions, excluding those in which any zero or infinite roots occur, is 33. F. S. Macaulay.

Solution by Proposer.

It is assumed that no identical relations exist between a, b, c, d which would make the number of solutions different from that of the general case. For example, if $b=3$, or if $d=0$, or if ax^3+bx^2+cx+d has a square factor, the number of solutions would be altered.

There are three cases to consider according as x, y, z are all unequal, or all equal, or two equal and the third different.

(a) Let x, y, z be all unequal.

Put $$\Sigma x=p, \quad \Sigma\frac{1}{x}=q, \quad 2\Sigma yz-\Sigma x^2=r;$$

so that $$4\Sigma yz=p^2+r.$$

Now $$x\left(\frac{1}{y}+\frac{1}{z}\right)(y-z)^2=x\left(\Sigma\frac{1}{x}-\frac{1}{x}\right)(\Sigma x^2-2\Sigma yz+2x\Sigma x-3x^2)$$
$$=-(qx-1)(3x^2-2px+r)$$
$$=ax^3+bx^2+cx+d \text{ from (1)};$$
$$\therefore\ (3q+a)x^3-(2pq+3-b)x^2+(2p+qr+c)x-(r-d)=0.$$

This equation has three different roots, viz., x, y, z;

$$\therefore\ p=\Sigma x=\frac{2pq+3-b}{3q+a},$$
$$q=\Sigma\frac{1}{x}=\frac{2p+qr+c}{r-d},$$
$$\tfrac{1}{4}(p^2+r)=\Sigma yz=\frac{2p+qr+c}{3q+a}.$$

Multiplying up the first two of these, we have

$$pq+ap=3-b,$$
$$2p+dq=-c,$$

which determine p and q, giving two solutions. Also to each value of p and q corresponds one value of r.

To each of the two solutions for p, q, r corresponds a cubic whose roots are the values of x, y, z. This cubic is

$$4qx^2(x-p)+(p^2+r)(qx-1)=0.$$

To each solution for p, q, r corresponds 6 solutions for x, y, z; and there are therefore 12 solutions of the original equations in which the roots x, y, z are all different.

(b) Let x, y, z be all equal.

In this case there are clearly 3 solutions, given, namely, by the roots of the equation $ax^3+bx^2+cx+d=0$.

(c) Let y and z be equal, and x different.

Then from (1) we have

$$ax^3+bx^2+cx+d=0,$$

giving 3 values for x.

Also from (2) or (3) we have, putting $y=z$,

$$y\left(\frac{1}{y}+\frac{1}{x}\right)(y-x)^2=ay^3+by^2+cy+d.$$

Dividing out by $y-x$, since $y-x \neq 0$, we have

$$y\left(\frac{1}{y}+\frac{1}{x}\right)(y-x)=a(y^2+yx+x^2)+b(y+x)+c.$$

This is a quadratic for y, not divisible by $y-x$, giving two values for y for each value of x. The values of z are the same as those of y.

There are, therefore, 6 solutions in which $y=z \neq x$; and similarly 6 more when $z=x \neq y$, and 6 more when $x=y \neq z$.

The total number of solutions of the equations (1), (2), (3), excluding those in which any zero or infinite roots occur, is therefore $12+3+18=33$; and these, and also the solutions involving zeros and infinities, can all be found by solving quadratics and cubics.

221. Note on Question 221 by W. S. Cooney.

Question 221 is only a particular case of Pascal's theorem, because if the sides be taken in the order $ABFDEC$, the intersections of BF and EC, FD and CA are collinear with the intersection of the parallel lines AB and DE. Therefore, etc.

235. *If* $\sum_{r=1}^{n-1} x_r^2 - \sum_{r=}^{n-2} x_r x_{r+1} = c$, *prove that* x_r^2 *is not greater than* $2cr\left(1-\frac{r}{n}\right)$; *and that if* $r<n-r$, *the limits of* $x_r x_{n-r}$ *are* cr *and* $-cr\left(1-\frac{2r}{n}\right)$.

F. S. Macaulay.

Solution by H. W. Lloyd Tanner.

We have
$$2c=\sum_{p=1}^{p=r-1}\{(p+1)x_p-px_{p+1}\}^2(p^2+p)^{-1}$$
$$+\frac{x_r^2}{r}+x_r^2-2x_rx_{r+1}+\ldots+2x^2_{n-1},$$

and
$$2c=\sum_{p=1}^{p=n-r-1}\{(p+1)x_{n-p}-px_{n-p-1}\}^2(p^2+p)^{-1}$$
$$+\frac{x_r^2}{n-r}+x_r^2-2x_rx_{r-1}+\ldots+2x_1^2,$$

as is seen by writing down the coefficients of any square and of any product of x on the right. Also the terms on the right of the two equations beginning in each case with x_r^2 together make up $2c$.

Therefore, by addition, $2c=\Sigma+x_r^2\left(\frac{1}{r}+\frac{1}{n-r}\right)$

where Σ is a sum of squares, each multiplied by a positive coefficient $(p^2+p)^{-1}$.

Hence $$x_r^2 \ngtr 2cr(n-r)/n.$$

The inequality holds unless all the squares in Σ vanish, viz.

$$\frac{x_1}{1}=\frac{x_2}{2}=\ldots=\frac{x_r}{r};\ \frac{x_r}{n-r}=\frac{x_{r+1}}{n-r-1}=\ldots=\frac{x_{n-1}}{1}.$$

Thus only one square (x_r^2) can be equal to the limit, and that only if the series

$$0,\ x_1,\ x_2,\ \ldots\ x_r;\ 0,\ x_{n-1},\ x_{n-2},\ \ldots\ x_r$$

are both in A.P., which implies further that all the x are of the same sign.

Solution by PROPOSER.

The given equation may be written

$$(0-x_1)^2+(x_1-x_2)^2+\ldots+(x_{n-2}-x_{n-1})^2+(x_{n-1}-0)^2=2c.$$

If x_r be supposed to remain fixed while all the other letters x vary, then the sum of the quantities $0-x_1$, x_1-x_2, ... $x_{r-1}-x_r$ is constant, being equal to $-x_r$; and therefore the sum of the squares is least when they are all equal, in which case each one is equal to $-x_r/r$.

Hence the least possible value for the sum of the first r squares in the above equation is x_r^2/r. Similarly the least value of the sum of the remaining squares is $x_r^2/(n-r)$.

But the least value of the whole cannot be greater than $2c$.

$$\therefore\ x_r^2\left(\frac{1}{r}+\frac{1}{n-r}\right)\not> 2c,\quad x_r^2\not> 2cr\left(1-\frac{r}{n}\right).$$

Again, if we suppose x_r and $x_{n-r}(r<n-r)$ to remain fixed and all the other letters x to vary, the least value of $(o-x_1)^2+\ldots+(x_{r-1}-x_r)^2$ is x_r^2/r, that of $(x_r-x_{r+1})^2+\ldots+(x_{n-r-1}-x_{n-r})^2$ is $(x_r-x_{n-r})^2/(n-2r)$, and that of

$$(x_{n-r}-x_{n-r+1})^2+\ldots+(x_{n-1}-0)^2 \text{ is } x^2_{n-r}/r.$$

Hence
$$\frac{x_r^2+x^2_{n-r}}{r}+\frac{(x_r-x_{n-r})^2}{n-2r}\not> 2c;$$

$$\therefore\ (x_r-x_{n-r})^2\left(\frac{1}{r}+\frac{1}{n-2r}\right)+\frac{2x_rx_{n-r}}{r}\not> 2c,$$

and
$$(x_r+x_{n-r})^2\left(\frac{1}{r}+\frac{1}{n-2r}\right)-2x_rx_{n-r}\left(\frac{1}{r}+\frac{2}{n-2r}\right)\not> 2c;$$

$$\therefore\ x_rx_{n-r}\not> rc,\text{ and } x_rx_{n-r}\not< -rc\left(1-\frac{2r}{n}\right).$$

236. *If the smallest prime factor of a number not greater than m^n be not less than m, then the number contains not more than $n-1$ prime factors.* E. HILL.

Solution by F. S. MACAULAY.

If the number not greater than m^n has n prime factors, let them be $a_1, a_2, \ldots a_n$. Their product is not greater than m^n, and no one is less than m. This is possible only if $a_1=a_2=\ldots=a_n=m$, otherwise the product must be greater than m^n. The number then cannot contain more than $n-1$ prime factors unless the number be m^n, and m is prime. The question should have read "less than m^n" instead of "not greater than m^n," in order to be quite exact.

237. *Show that if an ellipse, parabola, and hyperbola have the same directrix and vertex, the ellipse will be entirely within the parabola, and the parabola entirely within the hyperbola.* T. ROACH.

Solution by T. ROACH.

If S_1, S, S_2 be the foci of the ellipse, parabola, and hyperbola, and, if possible, let P be a common point for the parabola and the ellipse or hyperbola. Writing S' for S_1 or S_2,

$$\frac{S'P}{PM}=\frac{S'A}{AX};\quad \frac{PN^2+S'N^2}{XN^2}=\frac{S'A^2}{AS^2};$$

$$\therefore\ AS^2(4AS.AN+AN^2+AS'^2-2AS'.AN)=AS'^2(AS^2+AN^2+2AS.AN);$$

$$\therefore\ AN^2(AS^2-AS'^2)=2AS.AS'^2.AN-4AS^3.AN+2AS^2.AS'.AN.$$

Whence $AN(AS-AS')=0$ or $AN(AS+AS')=-2AS(AS'+2AS)$;

therefore, whether AS' be $>$ or $<$ AS, $AN=0$ or is negative and impossible, therefore the vertex is the only point of meeting.

Mr. KNAPTON sends an analytical solution.

Solution by H. G. MAYO.

Let S, S' be foci of parabola and hyperbola, V common vertex, X the foot of the common directrix; Q a point common to the curves; Qq, QN perpendiculars to directrix and axis.

Then $$SQ>S'Q; \therefore SN>S'N$$

and $$SV<S'V \text{ for } \frac{SV}{VX}<\frac{S'V}{VX};$$

$$\therefore SN-S'V>S'N-SV, \text{ i.e. } SN-S'N>S'V-SV,$$

or $SS'<SS'$, which is absurd.

Q cannot lie to the right of S', for then would $SQ>S'Q$. Similarly for the ellipse.

238. *The area of any plane triangle whose sides and area are integers is divisible by* 6. ARTEMAS MARTIN.

Solution by W. J. GREENSTREET.

If $ab(c^2+d^2)$, $cd(a^2+b^2)$, $(ad+bc)(ac-bd)$ be the sides of any triangle, its area $\Delta=abcd(ad+bc)(ac-bd)$.

If a, b, c, d are each odd, $ad+bc$ and $ac-bd$ are both multiples of 2.

If a, b, c, d are not multiples of 3, each is of form $3x\pm1$;

$\therefore$ $(ad+bc)$ is of the form $3y\pm1\pm1$, and $ac-bd$ is of form $3z\pm1\mp1$;

$\therefore$ $ad+bc$ or $ac-bd$ is a multiple of 3; $\therefore$ Δ is a multiple of 6.

239. *In the process of converting* 1/100103 *into a repeating decimal the* 273rd *remainder is* 100067, *the* 5005th *is* 23447, *the* 6126th *is* 14, *the* 6290th *is* 26789, *and the* 7587th *is* 1000091. *Find where the remainders* 1, 2, 3, ... 10 *will occur.* (*A correction of* 149.) C. N. MURTON.

Solution by H. W. LLOYD TANNER.

The 'r^{th} remainder' is the remainder when 10^r is divided by p $(=100103)$, for in treating this remainder as an integer we consider the remainder, not of $1/p$, but of $10^r/p$. Thus the second datum may be written $10^{5005}\equiv23447$, whence in succession $10^{50050}\equiv70072$, $10^{50051}\equiv700720\equiv-1$, $10^{100102}\equiv1$ (which is Fermat's theorem). Three other data yield $10^{273}\equiv100067\equiv-36\equiv-2^2.3^2(\alpha)$; $10^{6126}\equiv2.7(\beta)$; $10^{7587}\equiv100091\equiv-2^2.3(\gamma)$. Hence using the values for 1, -1 already found, we have (α/γ), $10^{92783}\equiv3$; and (γ^2/α), $10^{64952}\equiv2^2$, so that $2\equiv10^{32476}$ or 10^{82527}. But since p is of the form $8n+7$, 2 is a quadratic residue of p, and the former value alone is right. From (β) $7=14/2\equiv10^{73752}$. Also $5=10/2\equiv10^{67627}$. The localisation of the remainders 6, 8, 9 now presents no difficulty. These results agree with those of Mr. Murton (pp. 116, 7). The unused datum (the 6290th remainder) is presumably tantamount to the quadratic residue property used above to discriminate between the two expressions for 2.

240. *If* α, β, γ ... *be random magnitudes subject to the condition that their sum is* S, *show that*

$$\epsilon(\alpha^p\beta^q\gamma^r\ldots)=\frac{\underline{|n-1}\,\underline{|p}\,\underline{|q}\,\underline{|r}\ldots}{\underline{|n-1+p+q+r+\ldots}}\,.\,s^{p+q+r+\ldots}$$

where p, q, r ... *are any integers and* $\epsilon(x)$ *denotes the expectation of* x.

W. ALLEN WHITWORTH.

Note by the Editor.

The proposer has proved this proposition in his pamphlet "The Expectation of Parts into which a Magnitude is divided at Random" (George Bell & Sons, 1898, 1/- net), p. 12, Prop. VI. Mr. Whitworth gives the following problems as applications of the foregoing theorem :

1. If a straight line be divided at random into any number of segments, the expectation of the cube on one of the segments is six times the expectation of the rectangular parallelopiped contained by three of the segments.

2. If unity be divided at random into n parts, the expectation of the continued product of the parts is

$$\frac{1}{n(n+1)(n+2)\ldots \text{ to } n \text{ factors}},$$

and the expectation of the r^{th} power of the continued product is

$$\frac{\{\underline{|r}\}^n}{n(n+1)(n+2)\ldots \text{ to } nr \text{ factors}}.$$

241. *Squares $BCA''A'$, $CAB''B'$, $ABC''C'$ are described on the sides of any triangle ABC (Δ). AA', BB', CC' intersect in P, Q, R; AA'', etc., in P', Q', R'; $C'A''$, etc., in Π_1, Π_2, Π_3;* $\Delta PQR = \Delta P'Q'R' = T$ (*v. E.T. Reprint*, vol. 58, p. 95); $\Delta\Pi_1\Pi_2\Pi_3 = S$; *the triangle formed by the whole lines AA', etc., is T'; and by $C''A''$, etc., is S'; $U = \Delta O_1O_2O_3$, where $O_1O_2O_3$ are the centres of the squares. Find T, T', S, S', U in terms of ω, the Brocard angle of Δ. Also show that PP', etc., intersect on the perpendiculars of Δ, and that the area V of the triangle formed by their intersections is given by*

$$V(2+\cot\omega)^2 = \Delta(1+\cot\omega)^2$$

or

$$V\Delta = (\Delta - T)^2.$$

W. S. Cooney.

Solution by Proposer.

Parallelograms $AB''A'''C'$, etc., are completed. A''', B''', C''' joined ; also A, A'', B'''. Then evidently sides of $A'''B'''C'''$, which pass through $O_1O_2O_3$, are equal respectively to $A'B''$, $B'C''$, $C'A''$, which pass through Z, X, Y, and $A'''B'''C''' = S'$, $AA''B''' = T'$.

Triangle $CC'''H = A'''B''H'$ $\therefore$ triangle $CC'''H +$ triangle $A'''H'A$

$= $ triangle $AA'''B'' = \Delta$ $\therefore$ triangle $A'''B'''C''' = \Delta + \frac{1}{2}(a^2+b^2+c^2) + 3\Delta$

$= 4\Delta + \frac{1}{2}(a^2+b^2+c^2) = 4\Delta + 2\Delta\cot\omega$;

$\therefore$ $S' = 2\Delta(2+\cot\omega)$.

But $S = 2T$, $S' = 2T'$, $TT' = \Delta^2$ (proved in *E. Times*), and $U = \frac{1}{4}S'$;

$$\therefore\ T = \frac{\Delta}{2+\cot\omega},\quad T' = \Delta(2+\cot\omega),$$

$$S = \frac{2\Delta}{2+\cot\omega},\quad U = \tfrac{1}{2}\Delta(2+\cot\omega).$$

Let $x =$ perpendicular from R' on AB, $y =$ perpendicular from P on BC, then

$$\frac{BR'}{BB''} = \frac{x}{B''E''} = \frac{x}{b\cos A}.$$

Similarly $\dfrac{y}{b\cos C}=\dfrac{BP}{BB'}$; but $\dfrac{BR}{BB''}=\dfrac{BP}{BB'}$, since PR' is parallel to $B'B''$;

$$\therefore\ \frac{x}{y}=\frac{\cos A}{\cos C};$$

$\therefore$ RR' and PP' meet on perpendicular from B on CA (since RR' and PP' are parallel to AB and BC respectively).

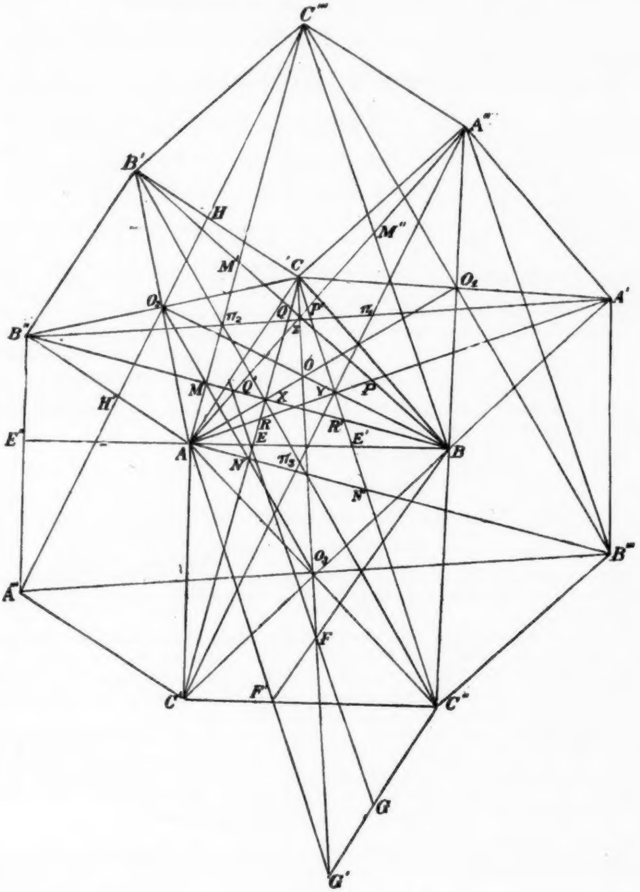


FIG. 1.

To find area of V.

$$\frac{x}{b\cos A}=\frac{BR'}{BB''};\ \text{but } Q'E \text{ is parallel to } CC'' \ (E.\ Times);$$

$$\therefore\ \frac{BR'}{R'Q}=\frac{BE'}{E'E}=\frac{BP''}{P''C}\ (P''\text{ is foot of perpendicular on }AB\text{ from }C)$$

$$=\cot B\quad \therefore\ BR'=R'Q'\cot B\,;\quad \therefore\ \frac{x}{b\cos A\cot B}=\frac{R'Q'}{BB''}$$

$$=\frac{\text{triangle }P'Q'R'}{\text{triangle }BP'B''}=\frac{T}{\Delta}=\frac{1}{2+\cot\omega}$$

(triangle $BPB''=CQ'C''=AR'A''=\Delta$, for $Q'E$, which is parallel to CC'', and BF' drawn parallel to CA meet COO_3G' in the same point F;

$$\therefore\ \square CG=\square CF'\,;\quad \therefore\ 2\text{ triangle }CQ'C''=2\Delta\,;\quad \therefore\ CQ'C''=\Delta.$$

Similarly $BP'B''=\Delta$, etc.).

But (see fig. 2) $b\cos A\cot B=DP''$, $D'P''=x$, where D is orthocentre;

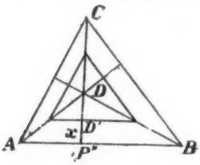


FIG. 2.

$$\therefore\ \frac{D'P''}{DP'}=\frac{1}{2+\cot\omega}\,;\quad \therefore\ \frac{DD'}{DP''}=\frac{1+\cot\omega}{2+\cot\omega}.$$

$$\frac{V}{\Delta}=\left(\frac{DD'}{DP''}\right)^2;\quad \therefore\ \frac{V}{\Delta}=\frac{(1+\cot\omega)^2}{(2+\cot\omega)^2}.$$

$$\text{Again, }\ \frac{V}{\Delta}=\left(1-\frac{1}{2+\cot\omega}\right)^2=\left(1-\frac{T}{\Delta}\right)^2;$$

$$\therefore\ V\Delta=(\Delta-T)^2.$$

242. *Given ABCD, a convex quadrilateral, such that $AB+CD=BC+AD$; show geometrically that a circle may be inscribed to the quadrilateral.*

R. F. DAVIS.

Solution by W. S. COONEY, J. L. THOMAS, F. MUIRHEAD.

Draw the circle touching BA, AD, DC, and if it does not touch CB, draw CB' touching it and cutting AB in B'.

Then $\qquad AB+DC=AD+BC$ (hyp.).

But $\qquad AB'+DC=AD+B'C$;

$$\therefore\ BC-B'C=AB-AB'=B'B,\ \text{which is impossible};\quad \therefore\ \text{etc.}$$

Solution by PROPOSER.

Produce BA, CD to meet in E, and describe the escribed circle to the triangle EAD (opposite to E) touching the sides of the triangle EAD in e, a, d respectively.

Then $Ae=Ad$, and $De=Da$; therefore $BC=Bd+Ca$. Take a point O upon BC such that $BO=Bd$; then $CO=Ca$.

Therefore B, C both lie on the radical axis of the point circle at O and the escribed circle. But this radical axis bisects the tangents from O to the escribed circle. Hence BC touches the escribed circle.

A very satisfactory and direct proof is given of this theorem in Nixon's *Euclid Revised* (3rd edition, p. 167). The above alternative proof is submitted as it leads to the following theorem: From two external points P, P' tangents PT, $P'T'$ are drawn to a given circle. If $PT\pm P'T'=PP'$, then PP' touches the circle (*Educational Times*, Question 13801).

243. *If ABCDE is a pentagon with given sides on a fixed base AB, and having equal angles at C and E, the locus of D is a circle.* A. C. DIXON.

Solution by J. A. THIRD.

Describe circles, centres A, B and radii $\sqrt{AE^2+ED^2}$, $\sqrt{BC^2+CD^2}$ respectively.

Let DA meet the first in P, P', and DB meet the second in Q, Q'.

Then

$$DP \cdot DP' = (DA + \sqrt{AE^2+ED^2})(DA - \sqrt{AE^2+ED^2}) = DA^2 - AE^2 - ED^2$$

$$= AE^2 + ED^2 - 2AE \cdot ED \cos E - AE^2 - ED^2 = -2AE \cdot ED \cos E.$$

So, $$DQ \cdot DQ' = -2BC \cdot CD \cos C;$$

$$\therefore\ DP \cdot DP'/(DQ \cdot DQ') = AE \cdot ED/(BC \cdot CD) = \text{constant}.$$

$\therefore$ locus of D is a circle coaxal with the fixed circles of the construction.

Solution by W. S. COONEY, R. F. DAVIS, E. P. BARRETT, C. F. SANDBERG.

Let $AD=x$, $BD=x'$, $A\hat{E}D = B\hat{C}D = \theta$, $AE=a$, $ED=b$, $DC=a'$, $CB=b'$.

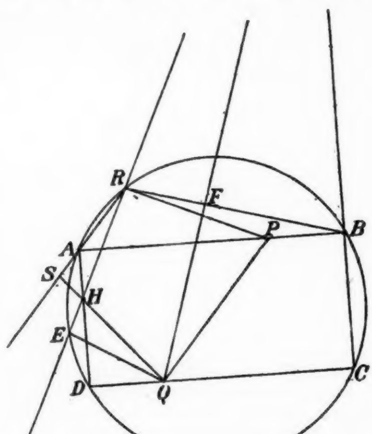
Then $$x^2 = a^2 + b^2 - 2ab\cos\theta\ ;\quad x'^2 = a'^2 + b'^2 - 2a'b'\cos\theta\ ;$$

$$\therefore\ nx^2 - mx'^2 = \text{const., where } m = ab,\ n = a'b'\ ;$$

$\therefore$ locus of D is a circle, centre K on AB produced, so that $AK/BK = n/m$.

244. *Through the centre of a rectangle ABCD any straight line is drawn, meeting AB in P and CD in Q. R is any point on the circum-circle of the rectangle, and the perpendiculars from Q on AR and BR meet AD and BC in H and K respectively. Prove that HK passes through R and is perpendicular to PR.*

W. J. DOBBS.



Solution by J. A. THIRD.

Through O the centre of the circle $ABCD$ draw OL parallel to QH, meeting PH in L. Then OL is perpendicular to AR, and therefore $AL = AR$.

Also OL bisects PH, for it passes through the mid point of PQ and is parallel to QH.

And PAH is a right angle;

$\therefore\ LP = LH = LA = LR$;

$\therefore\ P, A, H, R$ are concyclic, and $PRH = 90°$.

Similarly $PRK = 90°$;

$\therefore$ etc.

Solution by W. S. COONEY.

If QE (fig.) be perpendicular to HR, and QH meet RA in S, then from the triangles HQE, HRS, we have $H\hat{Q}E = H\hat{R}S$.

Also H, Q, D, E are concyclic; $\therefore\ H\hat{R}S = H\hat{Q}E = H\hat{D}E$;

. E is on the circle $ABCD$.

Also $ERFQ$, $QFBC$, $ERBC$ are cyclic quadrilaterals;

$\therefore\ ER, QF, CB$ meet in k the radical centre of the three circles; $\therefore$ etc.

Solution by R. F. Davis, J. C. Palmer.

Let KQ cut BR in N.

Since $A\hat{R}C = B\hat{R}D$ = a right angle, by similar triangles

$QC : RM = AH : MH$

or $AP : AH = RM : MH$, therefore $A\hat{P}H = M\hat{R}H$.

Thus $ARHP$ is cyclic, and $P\hat{R}H$ a right angle.

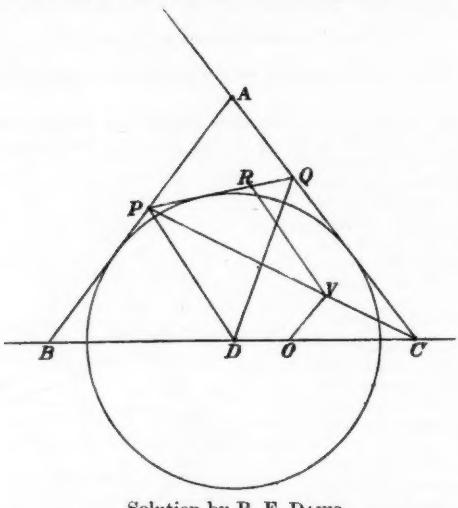
Again $QD : RN = BK : KN$

or $PB : BK = RN : KN$, therefore $B\hat{P}K = B\hat{R}K$.

Thus $BPRK$ is also cyclic, and $P\hat{B}H$ a right angle.

Therefore H, K are collinear with R.

245. *Given the perimeter of a triangle and the position of two sides. Find by elementary geometry the locus of a point dividing the base in a given ratio.* J. Elliott.



Solution by R. F. Davis.

Let ABC be an isosceles triangle. With D the mid-point of the base BC as centre describe a circle touching the equal sides AB, AC. Let PQ be any tangent to this circle forming with AB, AC a variable escribed triangle APQ whose perimeter is constant and whose sides AP, AQ have fixed positions AB, AC.

Then the triangles BPD, CDQ are similar, having their angles equal to the complements of the semi-angles of APQ.

Hence $BP \cdot CQ = BD \cdot CD$ = constant.

If R divides PQ in a fixed ratio and $PV : VC = BO : OC = PR : RQ$, then O is a fixed point and $OV \cdot VR$ = constant. The locus of R is therefore a hyperbola having its centre at O, and its asymptotes parallel to AB, AC.

246. *If the sum of the plane angles at each vertex of a tetrahedron be two right angles, prove that the opposite edges are equal.* E. Fenwick.

Solution by J. B. Parish, R. F. Davis, J. C. Palmer.

Let $ABCD$ be the tetrahedron, and A_1, A_2, A_3 the positions of A when the three faces are turned round CD, DB, BC so as to lie in the plane of BCD.

Then A_1DA_2, A_2BA_3, A_3CA_1 are straight lines and BCD is the median triangle of $A_1A_2A_3$; $\therefore$ the four faces are equal and similar.

247. *CP, CP', CD, CD' are four straight lines. If (three letters denoting an area) $CPD = CP'D'$, $CPD' = CP'D$, and $CPP' = CDD'$; then the lines are pairs of conjugate radii of an ellipse.* R. W. Genese.

Solution by F. S. Macaulay.

Let CP, CP', CD, CD' be arranged within an angle less than two right angles. This does not affect the problem, since the areas of the triangles mentioned are taken positively. Also let CP, CP', CD, CD' be the order of the four rays in rotation. The figure can then be projected orthogonally into cp, cp', cd, cd' so that the angles pcd, $p'cd'$ are both right angles.

Let $\qquad cp=p,\ \ cp'=p',\ \ cd=d,\ \ cd'=d'\ \ p\hat{c}p'=\theta.$

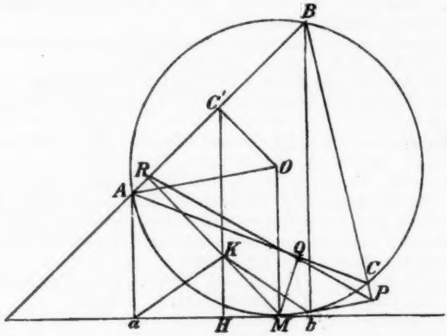
Then $\qquad \Delta cpd=\Delta cp'd'$ gives $pd=p'd'$;

$\qquad \Delta cpd'=\Delta cp'd$ gives $pd'\cos\theta=p'd\cos\theta$;

and $\qquad \Delta cpp'=\Delta cdd'$ gives $pp'\sin\theta=dd'\sin\theta$.

We have therefore $pd=p'd'$, $pd'=p'd$, $pp'=dd'$, and thence $p=p'=d=d'$. Thus cp, cp', cd, cd' are pairs of conjugate radii of a circle, and CP, CP', CD, CD' are similar pairs for an ellipse. The pairs which are conjugate are the first and third in the order of rotation, and the second and fourth, all four radii lying in the same half of a plane.

248. *If P, Q, R be the Simson line of a point M on the circum-circle of the triangle ABC, and a, b, c the feet of the perpendiculars from vertices on the tangent at M, then the circles bcP, caQ, abR are respectively tangential to the sides of ABC.* W. J. Greenstreet.

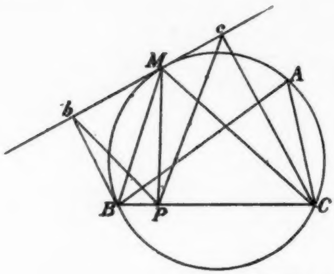


Solution by J. A. Third.

Join O, C' the circumcentre and mid point of AB. Let $C'H$ the perpendicular on Ma cut RM in K. Join aK, bK, OA. Let BA, MHa intersect at $\angle\theta$. Then $M\hat{K}H=C'\hat{K}R=\theta$. $OMKC'$ is a parallelogram;

$$\therefore\ AO=OM=C'K;\ aK^2=Ha^2+HK^2=C'A^2\cos^2\theta+KM^2\cos^2\theta$$
$$=\cos^2\theta(C'A^2+C'O^2)=C'K^2\cos^2\theta=RK^2.$$

Again, $C'H$ bisects AB, and is perpendicular to ba; $\therefore\ bK=aK=KR$;
$\therefore$ KR is a radius of the circle abR and is perpendicular to AB;
$\therefore$ the circle abR touches AB at R; whence, etc.



Solution by W. S. Cooney, J. V. Thomas, and Proposer.

From the cyclic quadrilaterals $bBPM$, $cCPM$ we have

$$b\hat{M}P = c\hat{C}P\ ;\quad B\hat{M}P = B\hat{b}P\ ;$$

and $$b\hat{M}B = M\hat{C}B\ ;$$

$\because$ bM is a tangent;

$$\therefore\ B\hat{M}P = c\hat{C}M = c\hat{P}M\ ;$$

$$\therefore\ c\hat{P}C = c\hat{b}P,$$

i.e., circle bcP touches BC at P.

Solution by R. F. Davis.

Observing that the circles upon MA, MB, MC respectively pass through (Q, R, a), (R, P, b), (P, Q, c) we have

$$B\hat{P}b = B\hat{M}b = P\hat{Q}M = P\hat{c}b\ ;$$

therefore BC touches the circle bcP.

Again supplement of $\quad C\hat{Q}c = C\hat{M}c = Q\hat{A}M = Q\hat{a}c\ ;$

therefore CA touches the circle caQ.

Lastly $\quad A\hat{R}a = A\hat{M}a = A\hat{B}M = R\hat{b}a\ ;$

therefore AB touches the circle abR.

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